NEW PROBLEMS RELATED TO THE VALENCES OF (SUPER) EDGE-MAGIC LABELINGS

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Abstract. A graph G of order p and size q is edge-magic if there is a bijective function $f: V(G) \cup E(G) \longrightarrow \{i\}_{i=1}^{p+q}$ such that $f(x)+f(xy)+f(y)=k$, for all $xy \in E(G)$. The function f is an edge-magic labeling of G and the sum k is called either the magic sum, the valence or the weight of f. Furthermore, if $f(V(G))=\{i\}_{i=1}^{p}$ then f is a super edge-magic labeling of G. In this paper we study the valences that can be attained by (super) edge-magic labelings of some families of graphs.

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1. Introduction

All graphs considered in this paper are simple graphs. For the graph theory terminology and notation not defined in this paper, we refer the reader to either [6] or [12].

Kotzig and Rosa ,[8], defined in 1970 the concepts of edge-magic graphs and edge-magic labelings as follows: let G be a $(p,q)-$ graph ($|V(G)|=p$ and $|E(G)|=q$). Then G is called *edge-magic* if there is a bijective function $f: V(G) \cup E(G) \longrightarrow \{i\}_{i=1}^{p+q}$ and $k \in \mathbb{N}$, such that $f(x)+f(xy)+f(y)=k$, for all $xy \in E(G)$. If such a function exists, then f is called an *edge-magic labeling* of G, and k is called the *valence* of f. Other common names for the valence of an edge-magic labeling are *edge-magic sum* and *edge-magic weight.* (See [2, 11]).

Motivated by the concept of edge-magic labelings, Enomoto et al. [3] introduced in 1998 the concept of super edge-magic labeling as follows: let G be a (p,q)-graph and let $f: V(G) \cup E(G) \longrightarrow \{i\}_{i=1}^{p+q}$ be an edge-magic labeling of G with the extra property that $f(V(G))=\{i\}_{i=1}^{p}$. Then G is called *super edge-magic* and f is called a *super edge-magic labeling* of G.

It is worthwhile mentioning that Acharya and Hegde introduced in [1] the concept of strongly indexable graph, that turns out to be equivalent to the concept of super edge-magic graph.

Figueroa-Centeno et al. provided in [4] the following useful characterization of super edge-magic graphs.

Lemma 1.1. [4] *Let G be a (p,q)-graph. Then G is super edge-magic if and only if there exists a bijective function $\bar{f}: V(G) \rightarrow \{i\}_{i=1}^{p}$, such that the set*

$$S(\bar{f})=\{\bar{f}(u)+\bar{f}(v): uv \in E(G)\}$$

is a set of q consecutive integers. In this case, $\bar{f}$ can be extended to a super edge-magic labeling of f.

We remark that the valence of the labeling f is determinated by the formula $p+q+\min S(\bar{f})$. From now on, we will call $\bar{f}$ the *canonical form* of the super edge-magic labeling f.

The following interesting conjecture, was introduced in [7] (see also research problem 25 in [11]) and as far as we know remains open in general.

Conjecture 1.2. [7]. *For $n = 2t+1 \geq 7$ and $5t+4 \leq j \leq 7t+5$ there is an edge-magic labeling of C_n, with valence $k = j$. For $n = 2t \geq 4$ and $5t+2 \leq j \leq 7t+1$ there is an edge-magic labeling of C_n, with valence $k = j$.*

Following a similar line of research, Figueroa-Centeno et al., proved in [5] the following result:

Theorem 1.3. [5] *The star $K_{1,n}$ is edge-magic. Furthermore, there are only three possible valences for edge-magic labelings of $K_{1,n}$. These valences are $2n+4$, $3n+3$ and $4n+2$. Moreover, only the first two valences correspond to super edge-magic labelings of $K_{1,n}$.*

Motivated by the previous result and by Conjecture 1.2, the concepts of perfect super edge-magic and perfect edge-magic graphs were introduced in [9] and in [10] respectively.

Let $G = (V, E)$ be a (p, q) graph. The set $S(G)$ is defined as follows:

$$S(G) = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{i=p+1}^{p+q} i}{q} : \text{ the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective} \right\}.$$

If $\lceil \min S(G) \rceil \leq \lfloor \max S(G) \rfloor$ then the *super edge-magic interval* of G, $I(G)$, is the set

$$I(G) = [\lceil \min S(G) \rceil, \lfloor \max S(G) \rfloor] \cap \mathbb{N}.$$

The *super edge-magic set* of G, $\sigma(G)$, is

$$\sigma(G) = \{k \in I(G) : \text{ there exists a super edge-magic labeling of } G \text{ with valence } k\}.$$

A graph G is said to be *perfect super edge-magic if* $I(G) = \sigma(G)$.

Now, the set $T(G)$ is defined as follows:

$$T(G) = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{e \in E} g(e)}{q} : g : V \cup E \to \{i\}_{i=1}^{p+q} \text{ is a bijective function} \right\}.$$

If $\lceil \min T(G) \rceil \leq \lfloor \max T(G) \rfloor$ then the *magic interval* of G, $J(G)$, is defined to be the set

$$J(G) = [\lceil \min T(G) \rceil, \lfloor \max T(G) \rfloor] \cap \mathbb{N}.$$

and the magic set of G, $\tau(G)$, is the set

$$\tau(G) = \{k \in J(G) : \text{ there exists an edge-magic labeling of } G \text{ with valence } k\}.$$

A graph G is said to be *perfect edge-magic if* $J(G) = \tau(G)$.

The following result was proved in [9]:

Theorem 1.4. [9] *Let C_m be a cycle of order $m = p^k$, where $p > 2$ is a prime number. Then the graph $G \cong C_m \odot \overline{K_n}$ is perfect super edge-magic for all $n \in \mathbb{N}$.*

The above result is interesting, since allows us to construct infinite families of perfect super edge-magic graphs with the property that the cardinality of the super edge-magic interval of the members of the family goes to infinity as we grow up in the family. That is to say, let $F_n \cong C_m \odot \overline{K_n}$ then $(F_1, F_2, \dots, F_n, \dots)$ is a family of super edge-magic graphs such that

$$\lim_{n\to+\infty} |I(F_n)| = +\infty.$$

Theorem 1.4 was generalized in [10] as follows:

Theorem 1.5. [10] *Let $m = p^k$, where p is an odd prime and $k \in \mathbb{N}$. Then the graph $G \cong C_m \odot \overline{K_n}$ is perfect edge-magic for all $n \in \mathbb{N}$.*

Motivated by the previous work, in this paper we define the concepts of $\mathfrak{F}^k$-family and $\mathfrak{E}^k$-family of graphs, as follows:

The infinite family of graphs $(F_1, F_2, \dots, F_n, \dots)$ is an $\mathfrak{F}^k$-*family* if

(i) each element F_n admits exactly k different valences for super edge-magic labelings, and
(ii) $\lim_{n\to+\infty} |I(F_n)| = +\infty$.

The infinite family of graphs $(F_1, F_2, \dots, F_n, \dots)$ is an $\mathfrak{E}^k$-*family* if

(i) each element F_n admits exactly k different valences for edge-magic labelings, and
(ii) $\lim_{n\to+\infty} |J(F_n)| = +\infty$.

From Theorem 1.3 it is clear that $(K_{1,2}, K_{1,3}, \dots, K_{1,n}, \dots)$ is an $\mathfrak{F}^2$-family and an $\mathfrak{E}^3$-family. The problems that will be studied in this paper are the following ones:

Problem 1.6. *For which $k \in \mathbb{N}$ is it possible to find $\mathfrak{F}^k$-families?. If it is possible to find such a family for some $k \in \mathbb{N}$, then construct the family.*

Problem 1.7. *For which $k \in \mathbb{N}$ is it possible to find $\mathfrak{E}^k$-families?. If it is possible to find such a family for some $k \in \mathbb{N}$, then construct the family.*

The goal of the next section is to study the existence of $\mathfrak{F}^k$-families and of $\mathfrak{E}^k$-families.

In order to conclude this introduction, we recall the concepts of complementary labelings of edge-magic graphs and of complementary labeling of the canonical form of a super edge-magic labeling. The complementary labelings of edge-magic labelings are also edge-magic labelings and the complementary labeling of the canonical form of a super edge-magic labeling is also the canonical form of a super edge-magic labeling itself. Let f be an edge-magic labeling of a (p,q)-graph G. Then, the *complementary labeling* of f, f^c, is defined by $f^c(u) = p + q + 1 - f(u)$, for each $u \in V(G) \cup E(G)$. Let f be a super edge-magic labeling of a (p,q)-graph G. Then, the complementary labeling of $\bar{f}$, the canonical form of f, is defined by $\bar{f}^{cs}(v) = p + 1 - \bar{f}(v)$, for each $v \in V(G)$. We denote by f^{cs} the super edge-magic labeling of G induced by $\bar{f}^{cs}$.

2. Main Results

For every $k, n \in \mathbb{N}$, define the graph F_n^k as follows: if $k \geq 2$ then

$$V(F_n^k) = \{v_l\}_{l=0}^{k} \cup \{v_0^i, v_k^i\}_{i=1}^{n} \cup \{v_l^j\}_{l=1,2,\dots,k-1}^{j=1,2,\dots,n-1}$$

and

$$E(F_n^k) = \{v_l v_{l+1}\}_{l=0}^{k-1} \cup \{v_0 v_0^i, v_k v_k^i\}_{i=1}^{n} \cup \{v_l v_l^j\}_{l=1,2,\dots,k-1}^{j=1,2,\dots,n-1}.$$

If $k = 1$ then $V(F_n^1) = \{v_0, v_1\} \cup \{v_0^i, v_1^i\}_{i=1}^{n}$ and $E(F_n^1) = \{v_0 v_1\} \cup \{v_0 v_0^i, v_1 v_1^i\}_{i=1}^{n}$. Thus, F_n^k is a tree of order $(k+1)n + 2$.

2.1 $\mathfrak{F}^k$ -families of graphs

This section is devoted to construct $\mathfrak{F}^k$ -families of graphs for different values of k. We begin by introducing the following lemma.

Lemma 2.8. *Let $k \in \mathbb{N}$. Then,*

(i) $\lim_{n\to+\infty} |I(F_n^k)| = +\infty$.
(ii) $|\sigma(F_n^k)| \leq k$, *for* $n \geq 2k+2$.

Proof. Let $g : V(F_n^k) \to \{1, 2, \ldots, (k+1)n+2\}$ be a bijective function. Then the corresponding element in $S(F_n^k)$ is given by:

$$\frac{\sum_{i=1}^{2(k+1)n+3} i + n\sum_{l=0}^{k} g(v_l)}{(k+1)n+1} = 2(k+1)n + 5 + \frac{1 + n\sum_{l=0}^{k} g(v_l)}{(k+1)n+1}.$$

Thus, the minimum occurs when $\sum_{l=0}^{k} g(v_l) = \sum_{l=1}^{k+1} l$, that is,

$$2(k+1)n + 5 + \frac{k+2}{2} + \frac{-k}{2(k+1)n+2},$$

and the maximum occurs when $\sum_{l=0}^{k} g(v_l) = \sum_{l=0}^{k}((k+1)n + 2 - l)$, that is,

$$3(k+1)n + 6 - \frac{k}{2} + \frac{k}{2(k+1)n+2}.$$

Hence, we obtain

$$\lim_{n\to+\infty} |I(F_n^k)| = \lim_{n\to+\infty} |[\lceil \min S(F_n^k)\rceil, \lfloor \max S(F_n^k)\rfloor] \cap \mathbb{N}| = +\infty,$$

which proves (i).

Let us prove (ii). Assume that $f : V(F_n^k) \cup E(F_n^k) \to \{1, 2, \ldots, 2(k+1)n+3\}$ is a super edge-magic labeling of F_n^k and let $\sum_{l=0}^{k} f(v_l) = an + b$, for some $a, b \in \mathbb{N} \cup \{0\}$ with $0 \leq b < n$. Then, the valence of f is given by:

$$\frac{\sum_{i=1}^{2(k+1)n+3} i + n\sum_{l=0}^{k} f(v_l)}{(k+1)n+1} = 2(k+1)n + 5 + \frac{an^2 + bn + 1}{(k+1)n+1}. \tag{1}$$

Thus, by considering the integer division

$$\frac{an^2 + bn + 1}{(k+1)n+1} = \frac{a}{k+1}n + \frac{1}{k+1}(b - \frac{a}{k+1}) + \frac{1 - (b - a/(k+1))/(k+1)}{(k+1)n+1}, \tag{2}$$

we have $1 - (b - a/(k+1))/(k+1) = 0$, which implies that

$$a = (k+1)(b - (k+1)). \tag{3}$$

By replacing (3) in (1) and (2), we get that the valence of f is given by

$$(k+1+b)n + 6. \tag{4}$$

Moreover, the inequalities $\sum_{l=1}^{k+1} l \leq \sum_{l=0}^{k} f(v_l) \leq \sum_{l=0}^{k}((k+1)n - 2 - l)$, which can be written as

$$(k+1)(k+2)/2 \leq an + b \leq (k+1)^2 n + (k+1)(2 - k/2)$$

imply, using (3) and the assumption that $b < n$, that $k+2 \leq b \leq 2k+1$. Therefore, using (4), the super edge-magic set of F_n^k, $\sigma(F_n^k)$, is contained in:

$$\{ni+6 : \ i = 2k+3, 2k+4, \dots, 3k+2\},$$

and the minimum sums of adjacent vertices in the corresponding canonical form of the super edge-magic labelings are contained in:

$$\{nj+3 : \ j = 1, 2, \dots, k\}.$$

□

Theorem 2.9. *There exists an $\mathfrak{F}^k$-family, for each $k = 1, 2, 3$.*

Proof. Notice that, by Theorem 1.1., we know that the family $\{K_{1,1}, K_{1,2}, \dots, K_{1,n}, \dots\}$ is an $\mathfrak{F}^2$-family.

For each $k = 1, 2, 3$, consider the family $\{F_{n_0}^k, F_{n_0+1}^k, \dots, F_n^k, \dots\}$, where $n_0 \geq 2k+2$. We will prove that $\{F_{n_0}^k, F_{n_0+1}^k, \dots, F_n^k, \dots\}$ is an $\mathfrak{F}^k$-family. Suppose first that f is a super edge-magic labeling of F_n^k with minimum induced sum $nj+3$ in the canonical form $\bar{f}$, for some $j = 1, 2, \dots, k$. An easy check shows that, the minimum induced sum of the canonical form of the complementary super edge-magic labeling of f, f^{cs}, is $n(k+1-j)+3$. Thus, since all caterpillars are super edge-magic, it follows that $\{F_{n_0}^k, F_{n_0+1}^k, \dots, F_n^k, \dots\}$ is an $\mathfrak{F}^k$-family, for $k = 1$ and $k = 2$.

For $k = 3$ we should find two super edge-magic labelings f and g with different valences, such that the valence of f^{cs} is diferent from the valences of f and g, and the valences of g and g^{cs} are the same. We will construct the canonical forms of these labelings. Let $\bar{f}$ be the labeling of F_n^3 that assigns the labels from 1 up to $4n+2$ to the vertices according to the following order of vertices:

$$v_0, v_1^1, v_1^2, \dots, v_1^{n-1}, v_2, v_1, v_2^1, v_2^2, \dots, v_2^{n-2}, v_3, v_2^{n-1}, v_3^1, v_3^2 \dots, v_3^n, v_0^1, v_0^2, \dots, v_0^n.$$

Then, the minimum sum of adjacent vertices in $\bar{f}$ is $n+3$. Thus, the minimum sum of adjacent vertices in $\bar{f^{cs}}$ is $3n+3$.

Let $\bar{g}$ be the labeling of F_n^3 that assigns the labels from 1 up to $4n+2$ to the vertices according to the following order of vertices:

$$v_0, v_1^1, v_1^2, \dots, v_1^{n-1}, v_2, v_3^1, v_3^2, \dots, v_3^n, v_0^1, v_0^2, \dots, v_0^n, v_1, v_2^1, \dots, v_2^{n-1}, v_3.$$

Then, the minimum sum of adjacent vertices in $\bar{g}$ is $2n+3$.

□

2.2 $\mathfrak{E}^k$-families of graphs

This section is devoted to construct $\mathfrak{E}^k$-families of graphs for different values of k. We begin by introducing the following lemma.

Lemma 2.10. *Let $k \in \mathbb{N}$. Then,*

(i) $\lim_{n\to+\infty} |J(F_n^k)| = +\infty.$
(ii) $|\tau(F_n^k)| \leq 2k+1$, *for* $n \geq 3k+3$.

Proof. Let $g : V(F_n^k) \cup E(F_n^k) \to \{1, 2, \ldots, 2(k+1)n+3\}$ be a bijective function. Then the corresponding element in $T(F_n^k)$ is given by:

$$\frac{\sum_{i=1}^{2(k+1)n+3} i + n\sum_{l=0}^{k} g(v_l)}{(k+1)n+1} = 2(k+1)n+5+\frac{1+n\sum_{l=0}^{k} g(v_l)}{(k+1)n+1}.$$

Thus, the minimum occurs when $\sum_{l=0}^{k} g(v_l) = \sum_{l=1}^{k+1} l$, that is,

$$2(k+1)n+5+\frac{k+2}{2}+\frac{-k}{2(k+1)n+2},$$

and the maximum occurs when $\sum_{l=0}^{k} g(v_l) = \sum_{l=0}^{k}(2(k+1)n+3-l)$, that is,

$$4(k+1)n+6-\frac{k}{2}+\frac{k}{2(k+1)n+2}.$$

Hence, we obtain

$$\lim_{n\to+\infty} |J(F_n^k)| = \lim_{n\to+\infty} |[\lceil \min T(F_n^k)\rceil, \lfloor \max T(F_n^k)\rfloor] \cap \mathbb{N}| = +\infty,$$

which proves (i).

Let us prove (ii). Assume that $f : V(F_n^k) \cup E(F_n^k) \to \{1, 2, \ldots, 2(k+1)n+3\}$ is an edge-magic labeling of F_n^k and let $\sum_{l=0}^{k} f(v_l) = an+b$, for some $a, b \in \mathbb{N}\cup\{0\}$ with $0 \le b < n$. Then, the valence of f is given by: $2(k+1)n+5+(an^2+bn+1)/((k+1)n+1)$. Thus, by considering the integer division

$$\frac{an^2+bn+1}{(k+1)n+1} = \frac{a}{k+1}n + \frac{1}{k+1}(b-\frac{a}{k+1}) + \frac{1-(b-a/(k+1))/(k+1)}{(k+1)n+1},$$

we have that $a = (k+1)(b-(k+1))$. Moreover, the inequalities $(k+1)(k+2)/2 \le an+b \le (k+1)(2(k+1)n+3-k/2)$ imply that $k+2 \le b \le 3k+2$. Therefore, the edge-magic set of F_n^k, $\tau(F_n^k)$, is contained in:

$$\{ni+6 : \ i = 2k+3, 2k+4, \ldots, 4k+3\}.$$

□

Theorem 2.11. *There exists an $\mathfrak{E}^k$-family, for each $k = 3, 4$ and 7.*

Proof. Notice that, by Theorem 1.1. we know that the family $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}, \ldots\}$ is an $\mathfrak{E}^3$-family.

For each $k = 1, 2, 3$, consider the family $\{F_{n_0}^k, F_{n_0+1}^k, \ldots, F_n^k, \ldots\}$, where $n_0 \ge 3k+3$. From the proof of Theorem 2.9, we can assume the existence of a super edge-magic labeling of F_n^k with valence $ni+6$, for each $i = 2k+3, 2k+4, \ldots, 3k+2$. We will prove that $\{F_{n_0}^k, F_{n_0+1}^k, \ldots, F_n^k, \ldots\}$ is an $\mathfrak{E}^{2k+1}$-family, when $k = 1, 3$ and an $\mathfrak{E}^4$-family, when $k = 2$. Suppose first that f is a super edge-magic labeling of F_n^k with valence $ni+6$, for some $i = 2k+3, 2k+4, \ldots, 3k+2$. An easy check shows that, the complementary (edge-magic) labeling of f, f^c, has valence $(6k+6-i)n+6$. Thus, if there is an edge-magic labeling of F_n^k with valence $(3k+3)n+6$, the result follows. We will show such a labeling for $k = 1$ and $k = 3$, whereas for $k = 2$ we will prove that such a labeling does not exist.

Case $k = 1$. Consider the function $f : V(F_n^1) \cup E(F_n^1) \to \{1, 2, \ldots, 4n + 3\}$ defined by

$$f(v) = \begin{cases} 1 & \text{if } v = v_0, \\ 4n+3 & \text{if } v = v_1, \\ 2n+2+i & \text{if } v = v_0^i, \\ 1+i & \text{if } v = v_1^i, \end{cases}$$

and $f(uv) = 6n + 6 - (f(u) + f(v))$, for each $uv \in E(F_n^1)$. An easy check shows that f is an edge-magic labeling of F_n^1 with valence $6n + 6$.

Case $k = 2$. Suppose that there exists an edge-magic labeling f of F_n^2 with valence $9n + 6$, in particular $f(v_0) + f(v_1) + f(v_0v_1) = 9n + 6$. From the proof of Lemma 2.10, it turns out that the sum $\sum_{l=0}^{2} f(v_l)$ is equal to $9n + 6$, which implies that $f(v_2) = f(v_0v_1)$, a contradiction. Therefore, such a labeling does not exist.

Case $k = 3$. Consider the function $f : V(F_n^3) \cup E(F_n^3) \to \{1, 2, \ldots, 8n + 3\}$ defined by

$$f(v_0) = 4n-1, f(v_1) = 4n+3, f(v_2) = 4n+1, f(v_3) = 4n+5, f(v_0^{n-1}) = 4n-3,$$

$$f(v_0^n) = 4n-2, f(v_1^{n-1}) = 4n-4, f(v_2^{n-1}) = 4n-6, f(v_3^{n-1}) = 4n-5, f(v_3^n) = 4n-7,$$

$$f(v) = \begin{cases} 4i & \text{if } v = v_0^i,\ 1 \le i \le n-2, \\ 4i-2 & \text{if } v = v_1^i,\ 1 \le i \le n-2, \\ 4i-1 & \text{if } v = v_2^i,\ 1 \le i \le n-2, \\ 4i-3 & \text{if } v = v_3^i,\ 1 \le i \le n-2, \end{cases}$$

and $f(uv) = 12n + 6 - (f(u) + f(v))$, for each $uv \in E(F_n^3)$. An easy check shows that f is an edge-magic labeling of F_n^3 with valence $12n + 6$.

□

In the next lines, we show that for each odd k, F_n^k admits an edge-magic labeling of valence $3(k + 1)n + 6$. First, we prove the next lemma.

Lemma 2.12. *Let k be an odd integer with $k \ge 5$ and let M be a matching of size $k+3$. Then, we can label each stable set with labels from 1 up to $k+3$, such that the induced sums of adjacent vertices are:*

$$\{4, 4, 6, 8, \ldots, 2 + 2k, 4 + 2k, 4 + 2k\}.$$

Proof. Let $V(M) = \{x_i, y_i\}_{i=1}^{k+3}$, $E(M) = \{x_iy_i\}_{i=1}^{k+3}$ and assume that each vertex x_i is labeled with i. We distinguish two cases.

Case $k \equiv 1 \pmod 4$. Consider the labeling $\alpha : \{y_i\}_{i=1}^{k+3} \to \{i\}_{i=1}^{k+3}$ defined by:

$$\alpha(y_1) = 3, \alpha(y_3) = 1, \alpha(y_2) = 4, \alpha(y_{k+3}) = k + 1,$$

$$\alpha(y_i) = \begin{cases} i-4, & \text{if } 6 \le i \le k+1 \text{ and } i \equiv 2 \pmod 4, \\ i+4, & \text{if } 4 \le i \le k \text{ and } i \equiv 0 \pmod 4, \\ i, & \text{if } 5 \le i \le k+2 \text{ and } i \text{ is odd.} \end{cases}$$

Case $k \equiv 3 \pmod 4$. Consider the labeling $\alpha : \{y_i\}_{i=1}^{k+3} \to \{i\}_{i=1}^{k+3}$ defined by:

$$\alpha(y_1) = 3, \alpha(y_3) = 1, \alpha(y_4) = 2, \alpha(y_{k+3}) = k + 1,$$

$$\alpha(y_i) = \begin{cases} i+4, & \text{if } 2 \le i \le k \text{ and } i \equiv 2 \pmod 4, \\ i-4, & \text{if } 5 \le i \le k+1 \text{ and } i \equiv 0 \pmod 4, \\ i, & \text{if } 5 \le i \le k+2 \text{ and } i \text{ is odd.} \end{cases}$$

Then, the induced sums of adjacent vertices are: $\{4,4,6,8,\ldots,2+2k,4+2k,4+2k\}$.

□

Lemma 2.13. *Let k be an odd integer with $k \geq 5$. Then F_n^k has an edge-magic labeling with valence $3(k+1)n+6$.*

Proof. Let $a \in \mathbb{Z}$ and $B \subset \mathbb{Z}$. We let $a+B=\{a+b: \ b \in B\}$. We will show an edge-magic labeling of F_n^k with valence $3(k+1)n+6$ that assigns the labels $(k+1)n+\{2-k,3-k,\ldots,2+k\}$ to the subgraph of F_n^k induced by $\{v_0,v_1,\ldots,v_k\}$ and the labels $\{1,2,\ldots,(k+1)n+1-k\}$ to the leaves.

Consider the subgraph G_n^k of F_n^k induced by $\{v_l\}_{l=0}^k \cup \{v_l^i\}_{l=0,1,\ldots,k}^{i=1,2,\ldots,n-2}$ and the injective function $g: V(G_n^k)\cup E(G_n^k) \to \{1,2,\ldots,2(k+1)n+3\}$ defined by:

$$g(v)=\begin{cases} (k+1)n+2-k+i & \text{if } v=v_i,\ i \text{ even},\\ (k+1)n+2+i & \text{if } v=v_i,\ i \text{ odd},\\ 1+(k-i)/2+(k+1)(l-1) & \text{if } v=v_i^l,\ i \text{ odd},\ 1\leq l\leq n-2,\\ (k+1)-i/2+(k+1)(l-1) & \text{if } v=v_i^l,\ i \text{ even},\ 1\leq l\leq n-2,\end{cases}$$

and $g(uv)=3(k+1)n+6-(g(u)+g(v))$, for each $uv \in E(G_n^k)$. Then

$$g(V(G_n^k)\cup E(G_n^k))=\{1,2,\ldots,2(k+1)n+3\}\setminus (X\cup Y),$$

where $X=(k+1)(n-2)+\{1,2,\ldots,k+3\}$ and $Y=(k+1)(n+1)+1+\{1,2,\ldots,k+3\}$. Note that, by Lemma 2.12, we can construct a matching with stable sets X and Y such that the induced sums are

$$(k+1)(2n-1)+1+\{4,4,6,8,\ldots,2+2k,4+2k,4+2k\}.$$

Let $f: V(F_n^k)\cup E(F_n^k) \to \{1,2,\ldots,2(k+1)n+3\}$ be the labeling that we construct by considering the extension of g defined by the previous matching as follows: we assign to v_0^{n-1} and v_0^n the labels x_0, x_0', where $x_0,x_0' \in X$ are incident in the matching to an edge with induced sum $(k+1)(2n-1)+5+2k$. We assign to v_k^{n-1} and v_k^n the labels x_k, x_k', where $x_k,x_k' \in X$ are incident in the matching to an edge with induced sum $(k+1)(2n-1)+5$. If a vertex v_i is labeled with $(k+1)n+2-k+2j$, for some $j=1,2,\ldots,k-1$ we assign to v_i^{n-1} the label x_j, where $x_j \in X$ is incident in the matching to an edge with induced sum $(k+1)(2n-1)+5+2k-2j$. Finally, we label the remaining edges with the labels in Y, in such a way that each vertex labeled with x_i is incident to the edge labeled with y_i. Hence, f is an edge-magic labeling of F_n^k with valence $3(k+1)n+6$.

□

Corollary 2.14. *Let $\mathcal{F}^k=(F_{n_0}^k,F_{n_0+1}^k,\ldots,F_n^k,\ldots)$, where $n_0 \geq 3k+3$. If $k\geq 5$ is odd and $\mathcal{F}^k$ is an $\mathfrak{F}^k$-family then $\mathcal{F}^k$ is an $\mathfrak{E}^{2k+1}$-family.*

Proof. As we have seen in the proof of Lemma 2.8, the super edge-magic set of $\mathcal{F}^k$ is contained in $\{ni+6: \ i=2k+3,2k+4,\ldots,3k+2\}$. Since we are assuming that $\mathcal{F}^k$ is an $\mathfrak{F}^k$-family, there exists a super edge-magic labeling f_i of F_n^k with valence $ni+6$, for each $i=2k+3,2k+4,\ldots,3k+2$. Thus, the complementary (edge-magic) labeling of f_i, f_i^c, has valence $(6k+6-i)n+6$, for each $i=2k+3,2k+4,\ldots,3k+2$. Hence, these labelings $\{f_i,f_i^c: \ i=2k+3,2k+4,\ldots,3k+2\}$ together with the labeling introduced in Lemma 2.13 imply that $|\tau(F_n^k)|\geq 2k+1$. Therefore, since by Lemma 2.10 we get $|\tau(F_n^k)|\leq 2k+1$, the result follows.

□

The careful reader would have already noticed that in order to prove Lemmas 2.8 and 2.10 we did not use any structural property of the graphs taken under consideration, other than the degree sequence of such graphs. Exactly the same conclusions of these two lemmas can be reached when considering graphs with the same degree sequences, and it is clear that the degree sequences of the graphs studied do not characterize the graphs. For instance, the family $\mathcal{G}^k = \{F_n^1 \cup (C_{k-1} \odot (n-1)K_1) : \ n \geq 3k+3\}$ has the required degree sequence, and therefore the conclusions of Lemmas 2.8 and 2.10 apply to it.

3. Other related problems

The main goal of this section is to introduce the concepts of valence density and super valence density for a graph G, that from now on will be denoted by $\delta(G)$ and by $\delta_s(G)$ respectively. Furthermore, we will extend these concepts to families of graphs. We will summarize what it is already known about these concepts, just by taking known results and restating them using the terminology introduced in this section. Finally, we will propose new problems and new conjectures that will open new lines of research.

Let G be a graph. The *valence density* of G, $\delta(G)$, is the quocient

$$\delta(G) = \frac{|\tau(G)|}{|J(G)|}.$$

The *super valence density* of G, $\delta_s(G)$, is the quocient

$$\delta_s(G) = \frac{|\sigma(G)|}{|I(G)|}.$$

Hence, it is a direct consequence of the definitions of $\delta(G)$ and $\delta_s(G)$ that $\{\delta(G), \delta_s(G)\} \subset [0,1]$. Also, G is a perfect edge-magic graph if and only if $\delta(G) = 1$ and G is a perfect super edge-magic graph if and only if $\delta_s(G) = 1$. On the other hand, it is clear that $\delta(G) = 0$ if and only if G is not edge-magic and $\delta_s(G) = 0$ if and only if G is not super edge-magic.

Let $\mathcal{F} = (F_1, F_2, \ldots, F_n, \ldots)$ be an infinite family of graphs. Then the *valence density of the family* $\mathcal{F}$, denoted by $\delta(\mathcal{F})$, is defined to be $\lim_{n\to+\infty} \delta(F_n)$, that is

$$\delta(\mathcal{F}) = \lim_{n\to+\infty} \frac{|\tau(F_n)|}{|J(F_n)|}.$$

The super valence density of the family $\mathcal{F}$, denoted by $\delta_s(\mathcal{F})$, is defined to be $\lim_{n\to+\infty} \delta_s(F_n)$, that is

$$\delta_s(\mathcal{F}) = \lim_{n\to+\infty} \frac{|\sigma(F_n)|}{|I(F_n)|}.$$

Next, just as a matter of completeness, we will introduce some easy remarks about these concepts.

Remark 2.15. *If G is a regular super edge-magic graph then $\delta_s(G) = 1$. If $\mathcal{F}$ is a family of regular super edge-magic graphs then $\delta_s(\mathcal{F}) = 1$.*

Remark 2.16. *It is well known (see* [11]*), for the cycle C_5, we have that $I(C_5) = \{14, 15, 16, 17, 18, 19\}$ and $\tau(C_5) = \{14, 16, 17, 19\}$. Therefore, $\delta(C_5) = 2/3$.*

Remark 2.17. *An easy check shows that* $J(K_{1,n}) = [2n+4, 4n+2]$ *and that* $I(K_{1,n}) = [2n+4, 3n+3]$. *Since by Theorem 1.3 we know that* $\tau(K_{1,n}) = 3$ *and* $\sigma(K_{1,n}) = 2$, *we get* $\delta(K_{1,n}) = 3/(2n-1)$ *and* $\delta_s(K_{1,n}) = 2/n$.

Remark 2.18. *Let* $m = p^k$, *where* p *is an odd prime and* $k \in \mathbb{N}$. *Consider the family* $\mathcal{F} = (F_1, F_2, \ldots, F_n, \ldots)$, *where* $F_n \cong C_m \odot \overline{K_n}$ *for all* $n \in \mathbb{N}$. *By Theorem 1.5, we get* $\delta(F_n) = 1$, *for all* $n \in \mathbb{N}$. *Therefore, we get* $\delta(\mathcal{F}) = 1$.

Remark 2.19. *Let* $m = p^k$, *where* p *is an odd prime and* $k \in \mathbb{N}$. *Consider the family* $\mathcal{F} = (F_1, F_2, \ldots, F_n, \ldots)$, *where* $F_n \cong C_m \odot \overline{K_n}$ *for all* $n \in \mathbb{N}$. *By Theorem 1.5, we get* $\delta_s(F_n) = 1$, *for all* $n \in \mathbb{N}$. *Therefore, we get* $\delta_s(\mathcal{F}) = 1$.

Remark 2.20. *Let* $\mathcal{F}^k = (F^k_{n_0}, F^k_{n_0+1}, \ldots, F^k_n, \ldots)$, *where* $n_0 \geq 3k+3$ *and* F^k_n *are the graphs introduced in Section 2, for* $k, n \in \mathbb{N}$ *and* $n \geq n_0$. *By Lemma 2.8 and Lemma 2.10 we get* $\delta_s(\mathcal{F}^k) = \delta(\mathcal{F}^k) = 0$, *for all* $k \geq 1$.

In order to conclude this section, we will formulate a conjecture (which is contained in Conjecture 1.2) and propose a set of open problems, that we feel that can be very interesting and challenging.

Conjecture 2.21. *Let* $\mathcal{F}$ *be the family of all cycles. Then*

$$\delta(\mathcal{F}) = 1.$$

Problem 2.22. *Let* $\mathcal{F}$ *be the family of all binary trees. Calculate* $\delta(\mathcal{F})$ *and* $\delta_s(\mathcal{F})$.

The *generalized Petersen graph* $P(n;k)$, $n \geq 3$ and $1 \leq k \leq \lceil (n-1)/2 \rceil$, consists of an outer n-cycle $x_0 x_1 \cdots x_{n-1} x_0$, a set of n-spokes $x_i y_i$, $0 \leq i \leq n-1$, and n inner edges of the form $y_i y_{i+_n k}$, where $+_n$ denotes the sum of two elements in the group $\mathbb{Z}_n$.

Problem 2.23. *Let* l *be any fixed positive integer and let* $n \in \mathbb{N}$ *such that* $\lceil (n-1)/2 \rceil \geq l$. *Let* $\mathcal{F}^l$ *be the family of generalized Petersen graphs of the form* $P(n;l)$. *Calculate* $\delta(\mathcal{F}^l)$. *In particular, calculate* $\delta(\mathcal{F}^l)$ *when* $l = 2$.

Finally, we feel that it would be interesting to study $\delta(G)$ and $\delta_s(G)$ when G is a caterpillar in particular, and a tree in general. Also when G is a unicyclic connected graph, since these families of graphs (caterpillars, trees, unicyclic connected graphs) have been the focus of many papers in graph labelings.

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